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ABSTRACT

The preconditioning technique has generally been accepted as an efficient procedure for accelerating the rate of convergence of an iterative method. One of the well-known examples is the preconditioned version of the conjugate gradient method for the solution of systems of linear equations. In this paper we study the application of the preconditioning technique for transonic flow problems, in which the governing equations are nonlinear and of mixed elliptic-hyperbolic type. Two iterative methods are presented, which are based on using a preconditioned conjugate gradient algorithm. Numerical results are reported which show that the present methods are computationally more efficient than the standard iterative procedure based on the successive line over-relaxation method.

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1. INTRODUCTION

The study of transonic aerodynamics has received considerable attention in the recent past, this is due to the fact that modern transport aircrafts operate at transonic speeds. The mathematical formulation of the transonic flow problem is well known, but its solution is not straightforward to obtain, because the governing partial differential equations are nonlinear and of mixed type. The standard numerical procedure for transonic flow calculations is based on the successive line overrelaxation method. However, not only does that method require an estimation of a relaxation parameter, but it also suffers from slow convergence as well. Alternative computational procedures such as the approximate factorization methods have been suggested recently [Holst and Ballhaus, 1979]. These methods have been shown to provide faster convergence rates than the successive line overrelaxation method if a set of iteration parameters is properly determined. Motivated by the difficulties in choosing optimal parameters a priori, it is therefore of strong interest to develop an efficient and reliable method which does not depend upon iteration parameters. Two such methods are presented in this paper based on using a preconditioning technique with the conjugate gradient method.

We discuss the basic mathematical formulation for the transonic flow problem in section 2, the standard relaxation method and the approximate factorization methods in section 3, two iterative procedures based on the preconditioned conjugate gradient algorithms in section 4, computational results in section 5, and finally, concluding remarks are given in section 6.

The purpose of this paper is to study the application of preconditioning techniques to accelerate the rate of convergence of iterative methods for transonic flow problems. Because of space limitations, many important aspects of the numerical solution for transonic flow calculations are not included.

2. TRANSONIC FLOW CALCULATIONS

2.1 Mathematical Formulation

The basic differential equation governing the flow of an inviscid, isentropic fluid is given by a kinematical relation representing the conservation of mass

$$\nabla\cdot\rho\vec{q} = 0 \qquad [1]$$

where

$$\vec{q} = \nabla\phi ,$$

$$\rho = (M_\infty^2 a^2)^{\frac{1}{\gamma-1}} ,$$

$$a^2 = \frac{1}{M_\infty^2} - \frac{\gamma-1}{2}(q^2-1) .$$

Here ϕ is the velocity potential, ρ, the density, M_∞, the Mach number at infinity, a, the local speed of sound and γ, the ratio of specific heats.

The tangential and wake boundary conditions, and the requirement that the velocity vanishes at infinity, complete the formulation of the governing equation.

For a two-dimensional flow in cartesian coordinates, Equation [1] can be expressed in the form

$$(\rho\phi_x)_x + (\rho\phi_y)_y = 0 \qquad [2]$$

This is known as the transonic full potential equation. In this paper we shall concentrate on the iterative solution of Equation [2]. From the mathematical point of view, the main difficulties associated with Equation [2] are as follows:

(a) the equation is nonlinear, where ρ is a function of ϕ_x and ϕ_y

(b) the equation is of mixed type: it changes from elliptic in subsonic regions to hyperbolic in supersonic regions, and the boundary between these regions is not known.

(c) the equation admits discontinuous solutions such as shocks which may exist in the flow.

(d) both compression and expansion shocks are admitted by the equation, and an additional condition must be introduced in order to eliminate the expansion shocks since they are physically meaningless.

For low Mach numbers ($M<1$) the flow is completely subsonic, and for high Mach numbers ($M>1$) the flow is completely supersonic (for airfoils with sharp leading and trailing edges). Figure 1 shows a typical transonic flow around an airfoil.

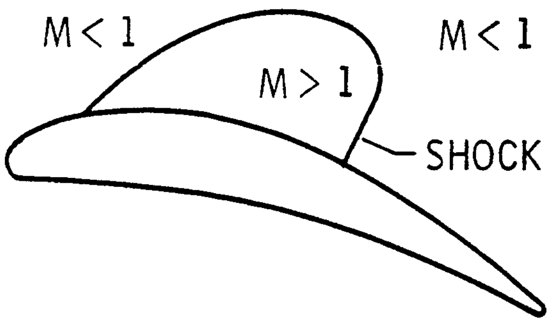

FIGURE 1. Transonic flow around an airfoil.

2.2 Numerical Procedures

It was first suggested by Murman and Cole [1971] that a type-dependent finite difference scheme could be used for transonic flow calculations. In their method central differences are used in subsonic regions, and upwind differences are used in the supersonic regions. It should be noted that an artificial viscosity is effectively introduced by using the upwind biased scheme in the supersonic regions, which in turn, is needed to eliminate the expansion shocks.

Another method, namely the artificial density method [Hafez, South and Murman, 1978], has recently been proposed where an artificial viscosity is easily implemented. In this method the transonic potential equation [2] is rewritten in the form

$$(\bar{\rho}\phi_x)_x + (\bar{\rho}\phi_y)_y = 0 \qquad [3]$$

where

$$\bar{\rho} = \rho - \mu\rho_s \Delta S \ ,$$

$$\mu = \max\,(0, 1 - a^2/q^2) \ .$$

Here ρ, a and q are defined as in Equation [1]. The term $\rho_s \Delta S$ is the product of the streamwise density gradient and the step length along a streamline. The use of $\bar{\rho}$ in Equation [3] produces a dissipative term when correct differencing is applied. It has been shown that retarding the density produces the same effect as the artificial viscosity introduced by the type-dependent schemes [Hafez, South and Murman, 1978].

Using the artificial density method, a central difference approximation can be applied to the modified equation [3] regardless of whether the region is subsonic or supersonic. An immediate consequence of this is that the regular structure of the matrix equation which results from the discretization of the linearized transonic equation is preserved. It should be noted that this property will be destroyed when an artificial viscosity is explicitly used. Because of this advantage the artificial density method is used in this paper.

3. ITERATIVE PROCEDURES

Assuming the density $\bar{\rho}$ is known from the previous iteration, a central difference approximation to Equation [3] is given by

$$(D_x^- \bar{\rho}_{i+\frac{1}{2},j} D_x^+ + D_y^- \bar{\rho}_{i,j+\frac{1}{2}} D_y^+)\ \phi_{i,j} = 0. \qquad [4]$$

where D_x^- and D_x^+ are the standard backward and forward difference operators in the x-direction, they are defined as follows.

$$D_x^- \phi_{i,j} = (\phi_{i,j} - \phi_{i-1,j})/\Delta x$$

$$D_x^+\phi_{i,j} = (\phi_{i+1,j} - \phi_{i,j})/\Delta x$$

Similar definitions are given for D_y^- and D_y^+.

The solution of the continuous problem is thus reduced to the solution of the following matrix equation

$$A(\phi^n)\ \phi^{n+1} = b \qquad [5]$$

where A is a symmetric positive definite matrix, whose elements are calculated from the previous solution ϕ^n.

A simple way to solve Equation [5] is by a first degree iterative scheme

$$\delta\phi^n = -\ \tau r^n \qquad [6]$$

where $\delta\phi^n = \phi^{n+1} - \phi^n$, $r^n = b - A(\phi^n)\ \phi^n$, and τ is an iteration parameter.

The simple iterative scheme [6] can be regarded as an iteration in pseudo-time, where the term $\delta\phi^n$ produces a time-step level of the scheme. If $\tau = \Delta t/\alpha$ then Equation [6] is a discretization for the equation

$$\alpha\phi_t = (\bar{\rho}\phi_x)_x + (\bar{\rho}\phi_y)_y \qquad [7]$$

Note that this scheme will converge for subsonic flow mainly. In order to obtain convergence for a transonic flow a ϕ_{xt} term must be included in the left hand side of Equation [7]. [Hafez and South, 1979].

In order to accelerate the convergence of the iterative scheme, a preconditioning matrix M, can be introduced. The preconditioned version of Equation [6] is

$$\delta\phi = -\ \tau M^{-1}\ r^n$$

or

$$M\delta\phi = -\tau r^n \qquad [8]$$

It is not hard to see that the matrix M determines the rate at which the iterative scheme converges. Generally speaking, if M is a good approximation of A, then a fast convergence rate can be expected.

To implement the preconditioned scheme [8] effectively, the matrix M must be easily invertible. A common approach is to factor M such that

$$M = M_1 M_2 \qquad [9]$$

and the product of the factors is an approximation to A. Furthermore, M_1 and M_2 should have a simple structure, such as a triangular matrix or a tridiagonal matrix.

We show that the successive line overrelaxation method and the approximate factorization methods can be described by the iterative scheme [8]. The main difference between these methods is in the choice of the preconditioning matrix M, and consequently, different rates of convergence result.

3.1 Successive Line Overrelaxation (SLOR) Method.

The SLOR method has been regarded as the standard iterative procedure for transonic flow calculations. The SLOR algorithm can be expressed by Equation [8], where

$$M\delta\phi_{i,j} = \left[\frac{1}{\Delta x}\left(-\frac{\rho_{i+\frac{1}{2},j}}{\Delta x} - \bar{\rho}_{i-\frac{1}{2},j} D_x^-\right) + \left(D_y^- \bar{\rho}_{i,j+\frac{1}{2}} D_y^+\right)\right]\delta\phi_{i,j} \qquad [10]$$

It should be noted that the scheme is semi-implicit, in the sense that it is explicit in the x-direction, and is implicit in the y-direction where an inversion of a tridiagonal matrix equation for a given value of i is required. Notice that $D_x^- \delta\phi_{i,j}$ in Equation [10] generates a ϕ_{xt} term, which is needed for the convergence in the supersonic regions.

Although the SLOR algorithm is reliable for transonic flow calculations, its convergence rate is slow for many practical problems. It is known that the rate of convergence of a fully implicit scheme can be faster than that of a semi-implicit or explicit iteration scheme. In the following, different preconditioning matrices M are studied, their choice will make the iterative scheme described in [8] become fully implicit.

3.2 Alternating Direction Implicit (ADI) Method.

Using the standard ADI scheme the matrix M can be constructed as follows

$$\alpha M\delta\phi_{i,j} = -(\alpha - D_x^- \bar{\rho}_{i+\frac{1}{2},j} D_x^+)(\alpha - D_y^- \bar{\rho}_{i,j+\frac{1}{2}} D_y^+)\ \delta\phi_{i,j} \qquad [11]$$

where α is an iteration parameter.

It is easily seen that the scheme is fully implicit. Multiplying the factors in Equation [11], it gives

$$\alpha M\delta\alpha_{i,j} = \alpha\left(-\alpha - \frac{D_x^- \bar{\rho}_{i+\frac{1}{2},j} D_x^+ D_y^- \bar{\rho}_{i,j+\frac{1}{2}} D_y^+}{\alpha} + D_x^- \bar{\rho}_{i+\frac{1}{2},j} D_x^+ + D_y^- \bar{\rho}_{i,j+\frac{1}{2}} D_y^+\right)\delta\phi_{i,j} \qquad [12]$$

Thus M is an approximation to the original problem, and the first two terms in the bracket represent the errors associated with the ADI scheme. It should be noted that there is no ϕ_{xt} term in this scheme, instead the first error term $\alpha\delta\phi_{i,j}$ produces a ϕ_t term in the iteration.

3.3 Approximate Factorization (AF) Method.

The AF method was first studied by Ballhaus and Steger [1975] for the transonic small disturbance equation, and it subsequently was applied to the transonic full potential equation by Holst and Ballhaus [1979]. In the AF scheme the matrix M is chosen as follows

$$\alpha M\delta\phi_{i,j} = -(\alpha D_x^- - D_y^- \bar{\rho}_{i,j+\frac{1}{2}} D_y^+)(\alpha - \bar{\rho}_{i+\frac{1}{2},j} D_x^+)\ \delta\phi_{i,j} \qquad [13]$$

Multiplying the factors of M gives

$$\alpha M\delta\phi_{i,j} = \alpha\left(-\alpha D_x^- - \frac{D_y^-\ \bar{\rho}_{i,j+\frac{1}{2}} D_y^+\ \bar{\rho}_{i+\frac{1}{2},j} D_x^+}{\alpha}\right.$$

$$\left. + D_x^-\ \bar{\rho}_{i+\frac{1}{2},j} D_x^+ + D_y^-\ \bar{\rho}_{i,j+\frac{1}{2}}\ D_y^+\right)\delta\phi_{i,j} \qquad [14]$$

It is important to note that unlike the ADI scheme the first error term $\alpha D_x^- \delta\phi_{i,j}$ in Equation [14] does produce a ϕ_{xt} term in the iteration.

In addition to the relaxation parameter τ associated with the iterative scheme [8], both ADI and AF methods require an estimation of the parameter α. It has been found that the high-frequency and low-frequency errors can be reduced effectively by choosing a suitable sequence for α.

Computational experiments show that the ADI scheme provides an excellent convergence rate for subsonic flow calculations. However, it is difficult to obtain convergence for cases of transonic flows. Divergence of the ADI scheme on nonuniform grids has been observed [South, 1981]. On the other hand, the AF scheme gives a good convergence rate for transonic flows provided the values of τ and α are properly chosen. The key success of the AF scheme is associated with the fact that a ϕ_{xt} term which is needed for convergence in the supersonic regions is included in this scheme.

It should be mentioned that another implicit approximate factorization scheme has recently been investigated by Sankar et al [1981], in which the factors of M are based on using the strongly implicit procedure [Stone, 1968]. Like the AF method the strongly implicit procedure also requires an estimation of a sequence of parameters α.

4. PRECONDITIONED CONJUGATE GRADIENT METHOD

In the previous section we have described several iterative procedures for transonic flow calculations. All of these methods can be regarded as using a preconditioning technique for a first-degree iterative scheme. The main difficulty in the application of these schemes efficiently is the requirement of choosing optimal parameters for τ and α. It should be pointed out that improper values for these parameters would lead to a slow convergence or even a divergence for the iterative scheme. To overcome this difficulty we shall now consider the method of conjugate gradients (CG).

The CG method was first proposed by Hestens and Stiefel [1952] for solving a symmetric and positive definite system of linear equation. The basic CG algorithm for the solution of Equation [5] is as follows.

Let ϕ^o be an arbitrary vector, compute

$$r^o = b - A\phi^o \quad , \; p^o = r^o$$

Then for n=0, 1, 2, ..., compute

$$\begin{aligned} \phi^{n+1} &= \phi^n + \alpha_n p^n \\ r^{n+1} &= r^n - \alpha_n A p^n \end{aligned} \qquad [15]$$

where

$$\alpha_n = (r^n, r^n)/(p^n, Ap^n)$$

$$p^{n+1} = r^{n+1} + \beta_n p^n$$

where

$$\beta_n = (r^{n+1}, r^{n+1})/(r^n, r^n)$$

Note that no estimation of iteration parameters is required in the CG algorithm. Other important properties and features of this method can be found in Reid [1971].

The CG algorithm given in Equations [15] can be rewritten in the following three-terms recurrence relation

$$(\phi^{n+2}-2\phi^{n+1}+\phi^{n})+(1-\frac{\alpha_{n+1}}{\alpha_n}\beta_n)(\phi^{n+1}-\phi^{n}) = \alpha_{n+1}r^{n+1}$$

or

$$\delta^2\phi + \omega\delta\phi = - \eta r^{n+1} \qquad [16]$$

Clearly it is a second-degree iterative scheme. If $\beta_n=0$ for all n, Equation [16] then reduces to a first degree scheme.

Over the last few years the preconditioning technique has been successfully applied with the CG algorithm for solving large sparse systems of linear equations [Evans, 1967; Axelsson, 1974; Meijerink and van der Vorst, 1976; Wong, 1979, etc.]. Let M be the preconditioning matrix, and consider the system

$$M^{-1} A\phi = M^{-1} b \qquad [17]$$

The preconditioned CG (PCG) algorithm for Equation [17] is as follows.

Let ϕ^o be an arbitrary vector, compute

$$r^o = b - A\phi^o ,$$

Solve $Mz^o = r_o$, Set $p^o = z^o$

Then for n = 0, 1, 2, ... compute

$$\phi^{n+1} = \phi^n + \alpha_n p^n$$

$$r^{n+1} = r^n - \alpha_n Ap^n$$

where

$$\alpha_n = (r^n, z^n)/(p^n, Ap^n) \qquad [18]$$

Solve $\quad Mz^{n+1} = r^{n+1}$

$$p^{n+1} = z^{n+1} + \beta_n p^n$$

where $\quad \beta_n = (r^{n+1}, z^{n+1})/(r^n, z^n).$

The PCG algorithm can be viewed as a preconditioning version of a second-degree iterative scheme, namely

$$M\ \delta^2\phi + \omega M\delta\phi = -\ \eta r^{n+1} \qquad [19]$$

Note that at each step in the PCG algorithm, the solution of the linear system Mz=r is required. In order to solve this matrix equation efficiently, M is usually chosen to be the product of M_1M_2 as defined in [9]. Different factorization for the matrix M lead to different PCG algorithms.

In this paper we shall consider two types of preconditioning matrices, in which M is an approximate factorization of A and $M=M_1M_2$.

(a) Row-sum agreement factorization [Wong, 1979]. In this factorization the following conditions are satisfied:

I) $(M_1)_{i,j}=(A_L)_{i,j}$ and $(M_2)_{i,j}=(A_U)_{i,j}$ where M_1 and M_2 are a lower and upper triangular matrices respectively, and M_1 and M_2 have nonzero elements only in those positions which correspond to the nonzero elements in the lower or upper triangular part of A.

II) For $A_{i,j}\neq 0$ and $i\neq j$, $M_{i,j}=A_{i,j}$ i.e. the off-diagonal elements of M whose locations correspond to the nonzero off-diagonal elements of A are set to those values.

III) The row-sums of M are the same as those of A.

(b) Symmetric successive overrelaxation (SSOR) preconditioning [Axelson, 1974].

Let A be written in the form

$$A = D - L - U$$

where D is a diagonal matrix, and L and U are strictly lower and upper triangular matrix respectively, then the preconditioning matrix M can be expressed by

$$M = \frac{1}{\omega(2-\omega)} \; (D-\omega L) \; D^{-1} \; (D-\omega U)$$

where ω is a relaxation parameter. The rate of convergence of the PCG method is not as sensitive as to choice of ω in the SOR iterative scheme.

Since the transonic potential equation [2] is nonlinear in nature, the solution of a nonlinear system of algebraic equations is required. The solution of this nonlinear problem can be obtained by solving a sequence of linearized systems of equations, however, the matrices generated by these linearized systems are different for each iteration step. In the problem considered here, the matrix equation is modified according to the solution obtained from the previous iteration. Consequently, the preconditioning matrix M will vary from iteration to iteration. The requirement for calculating the factorizations of M at each iteration step will increase the computational work considerably; thus, it would be very advantageous to choose a preconditioning matrix independent of the iteration step.

Recall that the matrix A in Equation [5] corresponds to the finite-difference approximation to

$$\partial_x(\bar{\rho}\,\partial_x) + \partial_y(\bar{\rho}\,\partial_y) \qquad [20]$$

Now consider another matrix B which results from the discretization for the following operator

$$\bar{\rho}\;(\partial_{xx} + \partial_{yy}) \qquad [21]$$

Let the preconditioning matrix M be an approximate factorization to B instead of A, with

$$M = M_1 M_2 M_3 \ , \qquad [22]$$

where M_1 is a diagonal matrix whose elements are given by $\bar{\rho}_{i,j}$, and the product of M_2M_3 is an approximation to the operator $(\partial_{xx}+\partial_{yy})$. The elements of M_2 and M_3 can be computed by either the row-sum agreement factorization or the SSOR preconditioning, and these values remain unchanged throughout the calculations.

The preconditioning based on [22] and [21] does not include a ϕ_{xt} term in the iterative scheme. However, if B is modified so that it is a difference approximation to

$$\bar{\rho}(\partial_{xx} + \partial_{yy} + \varepsilon\partial_x) \ , \qquad [23]$$

where ε is a parameter to be discussed shortly. Then a ϕ_{xt} term is introduced. The preconditioning matrix M is defined in Equation [22], where M_1 is the diagonal matrix denoting the density elements at each grid point, the product M_2M_3 is an approximate factorization to $(\partial_{xx} + \partial_{yy} + \varepsilon\partial_x)$.

In this paper two iterative procedures based on PCG methods are investigated.

(a) SLOR + CG scheme

This is a combination of the SLOR scheme and the PCG algorithm, the preconditioning matrix is based on [21] and [22]. The relaxation step is needed for convergence in cases of transonic flows because the PCG algorithm with the factorization given in [21] and [22] does not include a ϕ_{xt} term.

The present combined iterative scheme is similar to that proposed by Jameson [1976], in which a relaxation scheme and a fast solver are used. However, there are many advantages in using the CG method over the fast solver routine:

I) There is no restriction on the grid in any direction in the computational domain.

II) Since only the overall convergence for the nonlinear problem is of interest, it is sufficient to use a small number of iterations in the PCG algorithm for the solution of the linearized system. This in turn provides a considerable saving in the computational work.

III) The PCG algorithm can be easily extended for three-dimensional problems.

(b) PCG scheme

With the modified approximate factorization [22] and [23], a ϕ_{xt} term is explicitly included. Consequently a convergence for this purely PCG scheme for subsonic and transonic flow calculations can be ensured.

Note that if the parameter ε in [23] is a non-zero constant, then a ϕ_{xt} term will be introduced in both subsonic and supersonic regions. The ϕ_{xt} term is needed for convergence in the supersonic region only; however, its presence in the subsonic regions may decrease the convergence rate of the iterative scheme. To overcome this problem, ε is chosen as follows

$$\varepsilon = \frac{-1}{\Delta x}\,\mu \qquad [24]$$

where

$$\mu = \max(0,\ 1 - \frac{1}{M^2}) \quad ,$$

$$M^2 = \frac{a^2}{q^2} \quad .$$

a^2 and q^2 are defined in Equations [1] and [2]. The parameter μ can be regarded as a switching function, such that ε is zero in subsonic regions but nonzero in supersonic regions. With this definition for ε, the factorization of M must be modified according to the development of the supersonic regions. However, it is sufficient to update the approximate factorization for every k step, and k=5 will be used in the numerical experiments reported in the next section.

5. COMPUTATIONAL RESULTS

In order to compare the performance of the PCG methods and the standard iterative procedure based on the SLOR method, two test problems for transonic potential calculations around a circular cylinder and NACA 0012 airfoil are examined. Particular attention is focused on the comparison of rates of convergence for subsonic and transonic flow problems. The CPU times in seconds on the CYBER-175 computer are also reported. In all cases the maximum residual, R_{max}, is used to measure the convergence of the iterative process.

Preconditioning techniques based on the SSOR method and the row-sum agreement factorization have been compared, and the convergence rate of the former is slower than that of the latter for all problems tested. Thus the results for the PCG method presented here will be based on the row-sum agreement factorization to determine the preconditioning matrix M.

5.1 Flow Calculations Around a Circular Cylinder

The SLOR and the combined SLOR + CG methods are compared. A 61x31 grid is used in all cases with a uniform mesh in the θ-direction and a 15 percent stretching in the r-direction. Figures 2 and 3 show the rates of convergence of the two methods for subsonic and transonic cases respectively. The pressure distributions around the circular cylinder are plotted in Figure 4. Figure 5 shows the rates of convergence for different values of ω, the relaxation parameter in the SLOR method. Note that the convergence for the combined SLOR + CG method is almost the same for a wide range of ω.

5.2 Flow Calculations Around NACA 0012 Airfoil at Zero Angle of Attack

A 61x31 grid is used in all calculations, and the Mach number

M_∞=0.6 corresponds to a subsonic flow, while M_∞=0.85 corresponds to a transonic flow. Figures 6 and 7 show a comparison of the SLOR and SLOR + CG methods for two different grid systems. In Figure 6 a highly stretched grid is used which corresponds to stretching the physical domain from $-\infty$ to $+\infty$ in the x-direction and from 0 to $+\infty$ in the y-direction. In contrast, Figure 7 corresponds to a finite physical domain, in which uniform grids are used in both x- and y-directions. It is clear that the combined SLOR + CG method is not sensitive to the stretched grids. Figure 8 shows the results for the purely PCG method. The corresponding pressure distributions are shown in Figure 9.

6. CONCLUSION

Iterative procedures for subsonic and transonic flow calculations have been studied in this paper. It appears that current existing procedures, which include the SLOR, ADI, AF methods, can be regarded as the application of a preconditioning technique to a first degree iterative scheme. All these methods require an estimation of iteration parameters in order to obtain an optimal convergence rate. On the other hand, we show that the PCG algorithm provides a second degree iterative scheme, in which no estimation of any parameter is needed. This particular feature provides the basis for an efficient and reliable iterative procedure. Even for the case of the SLOR + CG scheme, the convergence of this combined method is not sensitive to the relaxation parameter as is the case for the SLOR scheme. For all problems tested here, the PCG methods provide faster rates of convergence than the SLOR method. They give excellent results for subsonic cases, and gives a modest improvement for transonic cases as well. It should be mentioned that the PCG methods have also been tested for transonic finite element calculations with similar success [Wong and Hafez, 1981]. It is important to note that the present PCG methods are not sensitive to the stretched grids, and their power increases when a finer grid is used and when a higher accuracy is required. It holds promise for 3D calculations as well.

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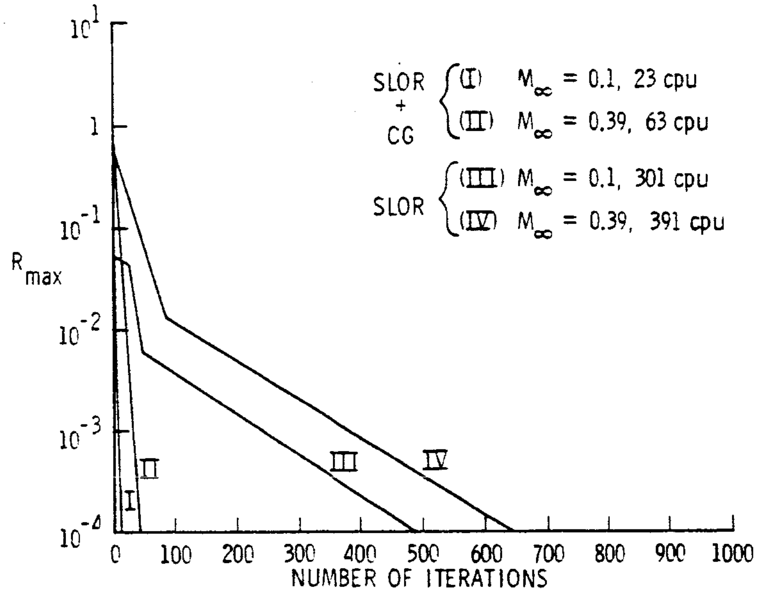

FIGURE 2. Rates of convergence for subsonic calculations around a circular cylinder.

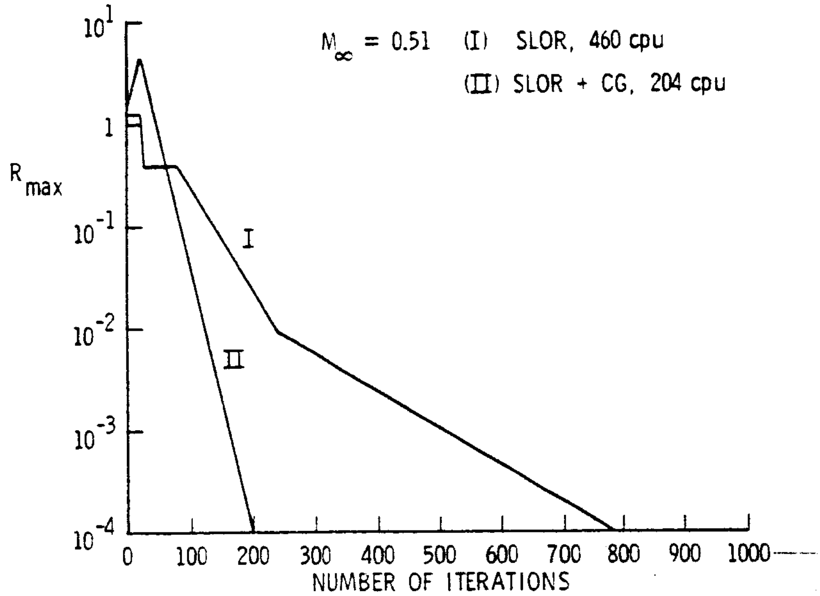

FIGURE 3. Rates of convergence for transonic calculations around a circular cylinder.

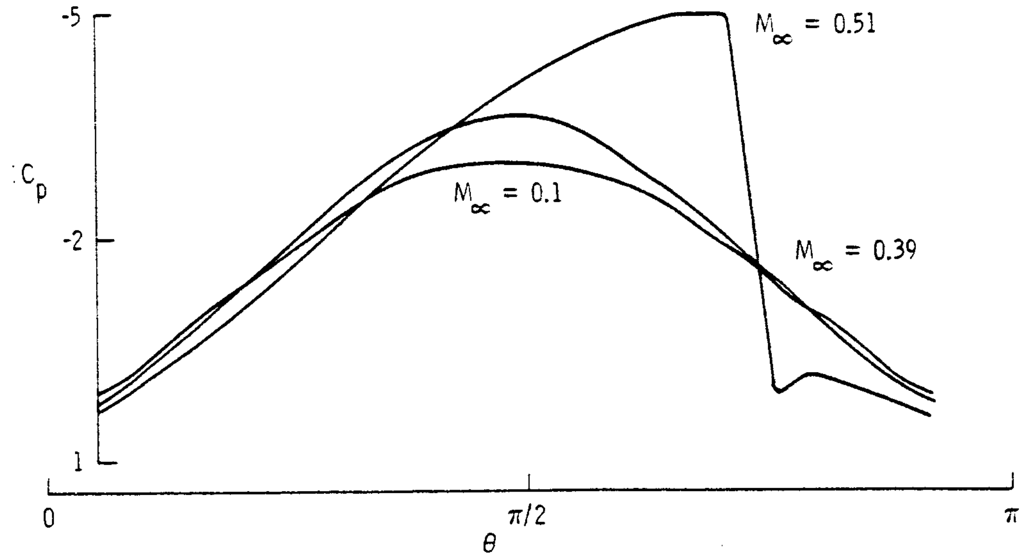

FIGURE 4. Pressure distributions around a circular cylinder at different Mach numbers.

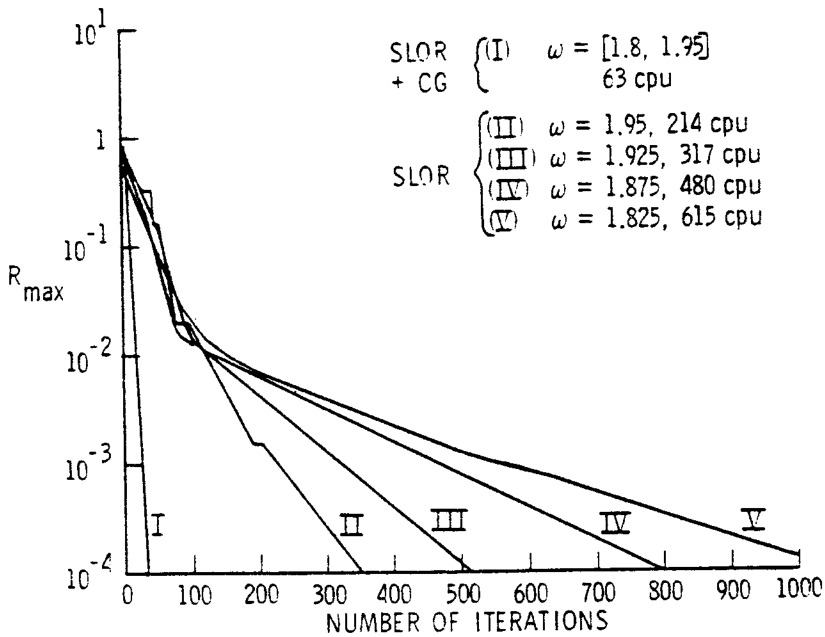

FIGURE 5. Dependence of convergence rates on relaxation parameter ω at $M_\infty = 0.39$.

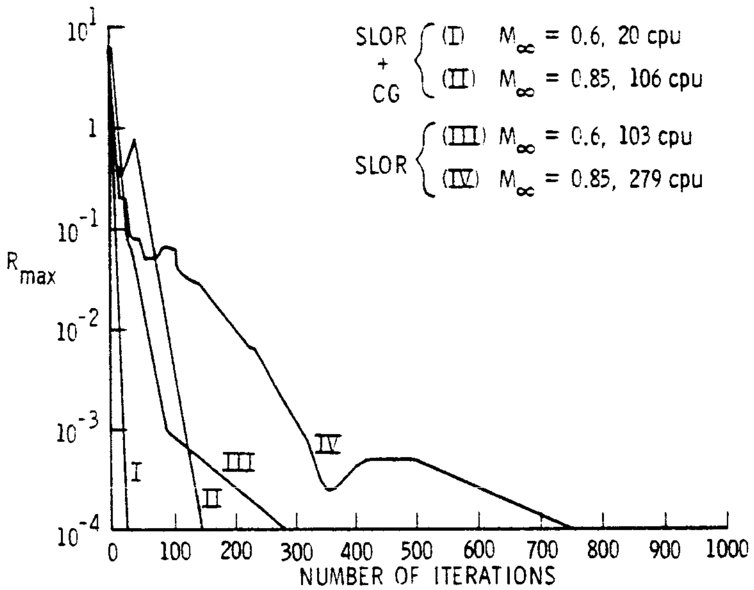

FIGURE 6. Rates of convergence for flow calculations around NACA 0012 airfoil with highly stretched grid.

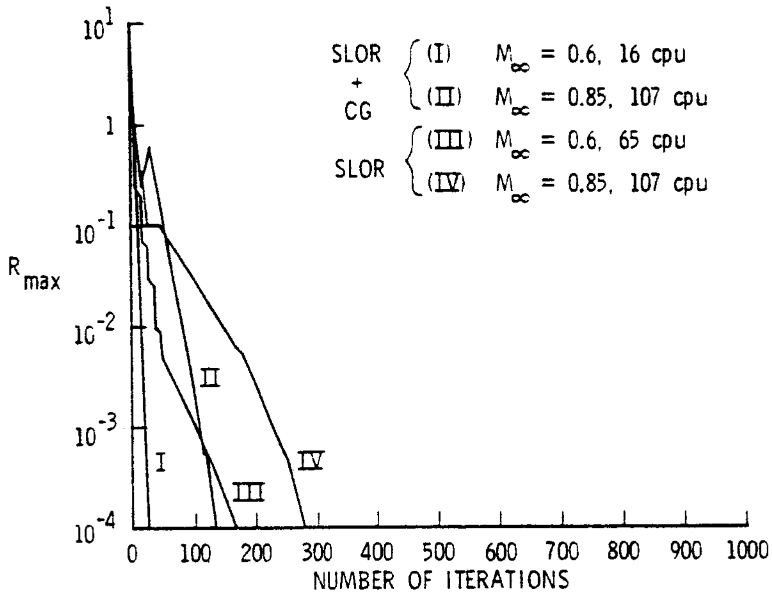

FIGURE 7. Rates of convergence for flow calculations around NACA 0012 airfoil with uniform grid.

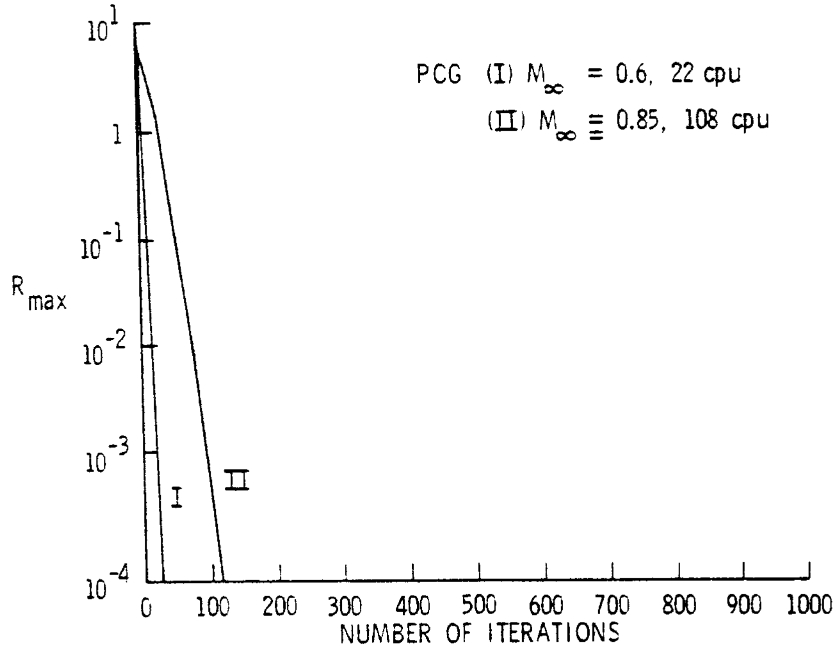

FIGURE 8. Rates of convergence for flow calculations around NACA 0012 airfoil.

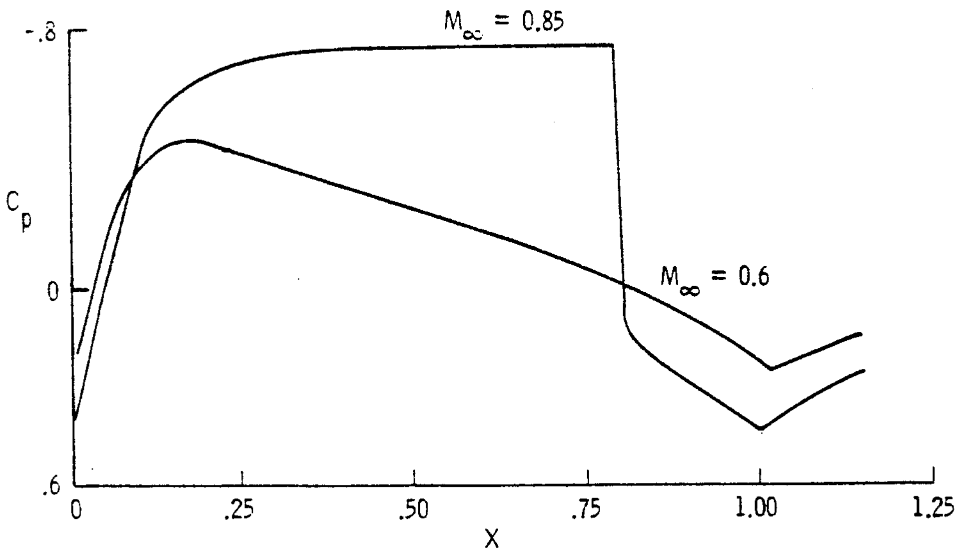

FIGURE 9. Pressure distributions around NACA 0012 airfoil at different Mach numbers.